

# LOWER BOUNDED SEMI-DIRICHLET FORMS ASSOCIATED WITH LÉVY TYPE OPERATORS

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**ABSTRACT.** Let  $k : E \times E \rightarrow [0, \infty)$  be a non-negative measurable function on some locally compact separable metric space  $E$ . We provide some simple conditions such that the quadratic form with jump kernel  $k$  becomes a regular lower bounded (non-local, non-symmetric) semi-Dirichlet form. If  $E = \mathbb{R}^n$  we identify the generator of the semi-Dirichlet form and its (formal) adjoint. In particular, we obtain a closed expression of the adjoint of the stable-like generator  $-(-\Delta)^{\alpha(x)}$  in the sense of Bass. Our results complement a recent paper by Fukushima and Uemura [3] and establishes the relation of these results with the symmetric principal value (SPV) approach due to Zhi-ming Ma and co-authors [5].

**Keywords:** non-local semi-Dirichlet forms; Lévy type operators; dual operators; stable-like processes

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Let  $(E, d, m)$  be a locally compact separable metric measure space. The reference measure  $m$  is a Radon measure with full topological support. Recently, Fukushima and Uemura [3] were able to construct a regular lower bounded semi-Dirichlet form and the corresponding jump-type Hunt process for a given jump kernel  $k(x, y)$ . One of the key ingredients in their construction are conditions that ensure that the symmetric part of the kernel,  $k_s$ , dominates the totally anti-symmetric part,  $k_a$ , where

$$k_s(x, y) := \frac{1}{2}(k(x, y) + k(y, x)) \quad \text{and} \quad k_a(x, y) := \frac{1}{2}(k(x, y) - k(y, x)).$$

For the readers' convenience let us briefly recall these assumptions, see [3, (2.1)–(2.4), Section 2],

$$(A0) \quad x \mapsto \int_{y \neq x} (1 \wedge d(x, y)^2) k_s(x, y) m(dy) \in L^1_{\text{loc}}(E, m),$$

$$(A1) \quad C_1 := \sup_{x \in E} \int_{d(x, y) \geq 1} |k_a(x, y)| m(dy) < \infty,$$

$$(A2) \quad C_2 := \sup_{x \in E} \int_{d(x, y) < 1} |k_a(x, y)|^\gamma m(dy) < \infty \quad \text{for some } \gamma \in (0, 1],$$

$$(A3) \quad C_3 := \sup_{x, y \in E, 0 < d(x, y) \leq 1} \frac{|k_a(x, y)|^{2-\gamma}}{k_s(x, y)} < \infty \quad \text{for } \gamma \text{ from (A2)}.$$

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In this short note we will simplify and generalize these conditions. If  $E = \mathbb{R}^n$  we will also show that the generator of the form and its formal adjoint is given by a symmetric Cauchy principal value integral (SPV) in the sense of Ma et al. [5]. This is motivated by and improves more recent development on non-local Dirichlet forms and Lévy type operators, e.g. [8, 3].

## 1. LOWER BOUNDED SEMI-DIRICHLET FORMS

Let  $(E, d, m)$  be a locally compact separable metric measure space equipped with a Radon measure  $m$ , and  $k(x, y)$  be a non-negative Borel measurable function on the space  $E \times E \setminus \Delta$ , where  $\Delta$  denotes the diagonal  $\{(x, x) : x \in E\}$  in  $E \times E$ . The inner product and the norm in  $L^2(E, m)$  are denoted by  $\langle \cdot, \cdot \rangle$  and  $\|\cdot\|_{L^2}$ , respectively. As before, denote by  $k_s$  and  $k_a$  the symmetric and totally anti-symmetric parts of  $k$ . Let  $C_0^{\text{Lip}}(E)$  be the space of uniformly Lipschitz continuous functions on  $E$  with compact support. Throughout this section, we will assume (A0).

A (not necessarily symmetric) bilinear form  $(\mathcal{E}, \mathcal{F})$ ,  $\mathcal{F} \subset L^2(E, m)$ , is a lower bounded Dirichlet form if the following conditions are satisfied. For some  $\gamma > 0$

- i)  $\mathcal{E}(u, u) \geq -\gamma \langle u, u \rangle$  for all  $u \in \mathcal{F}$ ;
- ii)  $\mathcal{E}(u, v) \leq c \sqrt{\mathcal{E}(u, u) + \gamma \langle u, u \rangle} \sqrt{\mathcal{E}(v, v) + \gamma \langle v, v \rangle}$  for all  $u, v \in \mathcal{F}$ ;
- iii)  $(\mathcal{F}, \mathcal{E}(\cdot, \cdot) + \gamma \langle \cdot, \cdot \rangle)$  is a complete subspace of  $L^2(E, m)$ ;
- iv)  $0 \wedge u \vee 1 \in \mathcal{F}$  for all  $u \in \mathcal{F}$  and  $\mathcal{E}(0 \wedge u \vee 1, u - 0 \wedge u \vee 1) \geq 0$ .

For further details we refer to [3, Section 1] and the references therein.

For each  $n \in \mathbb{N}$ , we define the operator  $L_n u$  for  $u \in C_0^{\text{Lip}}(E)$  by

$$L_n u(x) := \int_{\{y \in E : d(x, y) > 1/n\}} (u(y) - u(x)) k(x, y) m(dy), \quad x \in E,$$

and the quadratic form  $\eta_n(u, v)$  for  $u, v \in C_0^{\text{Lip}}(E)$  by

$$\eta_n(u, v) := -\langle L_n u, v \rangle = - \int_E L_n u(x) v(x) m(dx).$$

Due to (A0), all integrals appearing in the definition of  $L_n$  and  $\eta_n$  are absolutely convergent. Finally, set

$$\begin{aligned} \mathcal{E}(u, v) &= \iint_{y \neq x} (u(x) - u(y))(v(x) - v(y)) k_s(x, y) m(dx) m(dy), \\ \mathcal{F}^r &= \left\{ u \in L^2(E, m) : u \text{ is Borel measurable and } \mathcal{E}(u, u) < \infty \right\}. \end{aligned}$$

The condition (A0) ensures that  $(\mathcal{E}, \mathcal{F}^r)$  is a symmetric Dirichlet form on  $L^2(E, m)$ , and  $\mathcal{F}^r$  contains the space  $C_0^{\text{Lip}}(E)$ . As usual,  $\mathcal{E}_1(u, u) = \mathcal{E}(u, u) + \|u\|_{L^2}^2$ , and we write  $\mathcal{F}^0$  for the  $\mathcal{E}_1$ -closure of  $C_0^{\text{Lip}}(E)$  in  $\mathcal{F}^r$ . In particular,  $(\mathcal{E}, \mathcal{F}^0)$  is a regular symmetric Dirichlet form on  $L^2(E, m)$ , cf. [4, Example 1.2.4].

Our main result is the following simple condition which guarantees that the limit of the forms  $\eta_n(u, v)$ ,  $n \rightarrow \infty$  exists, and defines a regular lower bounded semi-Dirichlet form. This generalizes and simplifies the earlier result by Fukushima and Uemura [3, Proposition 1 and Theorem 1].

**Theorem 1.1.** *Assume that (A0) is satisfied and that*

$$(1.1) \quad \sup_{x \in E} \int_{\{k_s(x,y) \neq 0\}} \frac{k_a(x,y)^2}{k_s(x,y)} m(dy) < \infty$$

*holds. Then we have the following two statements.*

(i) *For all  $u, v \in C_0^{\text{Lip}}(E)$ , the limit  $\eta(u, v) = \lim_{n \rightarrow \infty} \eta_n(u, v)$  exists. The form  $\eta(u, v)$  has the following integral representation*

$$(1.2) \quad \eta(u, v) = \frac{1}{2} \mathcal{E}(u, v) + \iint_{y \neq x} (u(x) - u(y)) v(y) k_a(x, y) m(dx) m(dy),$$

*where the integral on the right hand side of (1.2) is absolutely convergent.*

(ii) *The form  $\eta$  extends from  $C_0^{\text{Lip}}(E) \times C_0^{\text{Lip}}(E)$  to  $\mathcal{F}^0 \times \mathcal{F}^0$  such that the pair  $(\eta, \mathcal{F}^0)$  is a regular lower bounded semi-Dirichlet form on  $L^2(E, m)$ .*

Let us briefly show that the conditions imposed by Fukushima and Uemura are stronger than (1.1). Indeed, if (A1)–(A3) hold, then we find for  $x \in E$ ,

$$\begin{aligned} & \int_{k_s(x,y) \neq 0} \frac{k_a(x,y)^2}{k_s(x,y)} m(dy) \\ & \leq \int_{\substack{d(x,y) \leq 1, \\ k_s(x,y) \neq 0}} \frac{|k_a(x,y)|^{2-\gamma}}{k_s(x,y)} |k_a(x,y)|^\gamma m(dy) + \int_{d(x,y) > 1} k_s(x,y) m(dy) \\ & \leq \left\{ \sup_{\substack{d(x,y) \leq 1, \\ k_s(x,y) \neq 0}} \frac{|k_a(x,y)|^{2-\gamma}}{k_s(x,y)} \right\} \int_{d(x,y) \leq 1} |k_a(x,y)|^\gamma m(dy) + \int_{d(x,y) > 1} k_s(x,y) m(dy) \\ & \leq C_2 C_3 + C_1. \end{aligned}$$

In the first inequality we have used that  $|k_a(x, y)| \leq k_s(x, y)$ .

*Sketch of the proof of Theorem 1.1.* From the definition of  $\eta_n$  we find for all  $u, v \in C_0^{\text{Lip}}(E)$  that

$$\eta_n(u, v) = \frac{1}{2} \mathcal{E}_n(u, v) + \iint_{d(x,y) \geq 1/n} (u(x) - u(y)) v(y) k_a(x, y) m(dx) m(dy),$$

where

$$\mathcal{E}_n(u, v) = \iint_{d(x,y) \geq 1/n} (u(x) - u(y))(v(x) - v(y)) k_s(x, y) m(dx) m(dy).$$

Because of (A0),  $\mathcal{E}_n(u, v)$  converges to  $\mathcal{E}(u, v)$  as  $n \rightarrow \infty$ .

To see the convergence of the non-symmetric part we set for  $x \in E$

$$h(x) := \int_{k_s(x,y) \neq 0} \frac{k_a(x,y)^2}{k_s(x,y)} m(dy).$$

An application of the Cauchy-Schwarz inequality and (1.1) show

$$\begin{aligned}
& \iint_{d(x,y) \geq 1/n} |u(x) - u(y)| |v(y)| |k_a(x, y)| m(dx) m(dy) \\
&= \iint_{\substack{d(x,y) \geq 1/n, \\ k_s(x,y) \neq 0}} |u(x) - u(y)| k_s(x, y)^{1/2} \times |v(y)| |k_a(x, y)| k_s(x, y)^{-1/2} m(dx) m(dy) \\
&\leq \left[ \iint_{d(x,y) \geq 1/n} (u(x) - u(y))^2 k_s(x, y) m(dx) m(dy) \right]^{\frac{1}{2}} \\
&\quad \times \left[ \iint_{\substack{d(x,y) \geq 1/n, \\ k_s(x,y) \neq 0}} v(y)^2 \frac{k_a(x, y)^2}{k_s(x, y)} m(dx) m(dy) \right]^{\frac{1}{2}} \\
&\leq \left[ \mathcal{E}_n(u, u) \right]^{1/2} \left[ \int v^2(y) h(y) m(dy) \right]^{1/2} \\
&\leq \left[ \mathcal{E}(u, u) \right]^{1/2} \|h\|_\infty^{1/2} \|v\|_{L^2}.
\end{aligned}$$

This shows that the expression

$$\iint_{d(x,y) \geq 1/n} (u(x) - u(y)) v(y) k_a(x, y) m(dx) m(dy)$$

converges absolutely as  $n \rightarrow \infty$  and (i) follows. In order to see (ii), we use (i) and the argument used in the proof of [3, Theorem 2].  $\square$

If the semi-Dirichlet form  $\eta(u, v)$  is given by (1.2), it is, in general, difficult to find a closed expression for its generator  $L$ . Recall that  $L$  satisfies for any  $u, v \in C_0^{\text{Lip}}(E)$

$$\eta(u, v) = -\langle Lu, v \rangle.$$

In the present situation this is possible if we assume that

$$(1.3) \quad \int_{0 < d(x,y) \leq 1} d(x, y) |k_a(x, y)| m(dy) < \infty, \quad x \in E.$$

Let

$$C^*(E) = \left\{ u \in C_0^{\text{Lip}}(E) : \text{SPV} \int_{y \neq x} (u(y) - u(x)) k_s(x, y) m(dy) \text{ exists} \right\},$$

where  $\text{SPV} \int \cdots dm$  means the symmetric Cauchy principle value in the sense of [5, Definition 2.5], i.e. for any increasing sequence  $(A_j)_{j \in \mathbb{N}}$  of relatively compact and symmetric sets  $A_j \subset E \times E$  such that  $\bigcup_{j \in \mathbb{N}} A_j = E \times E \setminus \Delta$ , the limit

$$\lim_{j \rightarrow \infty} \int_{A_j} (u(y) - u(x)) k_s(x, y) m(dy)$$

exists and is independent of the sequence  $(A_j)_{j \in \mathbb{N}}$ .

**Proposition 1.2.** *Assume that (A0), (1.1) and (1.3) hold. Then,*

$$\eta(u, v) = -\langle Lu, v \rangle, \quad u \in C^*(E), \quad v \in C_0^{\text{Lip}}(E),$$

where

$$Lu(x) = SPV \int_{y \neq x} (u(y) - u(x)) k_s(x, y) m(dy) + \int_{y \neq x} (u(y) - u(x)) k_a(x, y) m(dy).$$

*Proof.* The condition (1.3) ensures that the operator  $L$  is well defined. According to the proof of Theorem 1.1, (A0) and (1.1) imply that for any  $u \in C^*(E)$  and  $v \in C_0^{\text{Lip}}(E)$ , the form  $\langle Lu, v \rangle$  is also well defined and finite.

On the other hand, under (A0) and (1.1), Theorem 1.1 shows that

$$\eta(u, v) = \frac{1}{2} \lim_{j \rightarrow \infty} \mathcal{E}_j(u, v) + \iint_{y \neq x} (u(x) - u(y)) v(y) k_a(x, y) m(dx) m(dy),$$

with

$$\mathcal{E}_j(u, v) = \iint_{A_j} (u(x) - u(y))(v(x) - v(y)) m(dx) m(dy)$$

and where  $(A_j)_{j \in \mathbb{N}}$  is an increasing sequence of relatively compact and symmetric sets such that  $\bigcup_{j \in \mathbb{N}} A_j = E \times E \setminus \Delta$ . Since  $k_s(x, y) = k_s(y, x)$ ,

$$\begin{aligned} \mathcal{E}_j(u, v) &= \frac{1}{2} \iint_{A_j} (u(x) - u(y)) v(x) k_s(x, y) m(dx) m(dy) \\ &\quad - \frac{1}{2} \iint_{A_j} (u(x) - u(y)) v(y) k_s(x, y) m(dx) m(dy) \\ &= \frac{1}{2} \iint_{A_j} (u(y) - u(x)) v(y) k_s(y, x) m(dx) m(dy) \\ &\quad - \frac{1}{2} \iint_{A_j} (u(x) - u(y)) v(y) k_s(x, y) m(dx) m(dy) \\ &= \iint_{A_j} (u(y) - u(x)) v(y) k_s(x, y) m(dx) m(dy), \end{aligned}$$

and the claim follows by the dominated convergence theorem.  $\square$

To get an expression for the generator associated with the semi-Dirichlet form  $\eta(u, v)$ , we need to characterize the domain  $C^*(E)$ . In the following we assume that  $E = \mathbb{R}^n$  is equipped with the Euclidean metric  $d(x, y) = |x - y|$  and Lebesgue measure  $m(dx) = dx$ .

**Theorem 1.3.** *Suppose that*

$$(1.4) \quad x \mapsto \int_{y \neq x} (1 \wedge |y - x|^2) k_s(x, y) dy \in L_{\text{loc}}^2(dx),$$

and

$$(1.5) \quad x \mapsto \int_{|y-x| \geq 1} k_s(x, y) dy \in L^2(dx) \cup L^\infty(dx).$$

If (1.1) and (1.3) hold, then the generator  $L$  associated with the semi-Dirichlet form  $\eta(u, v)$  given by (1.2) is of the following form

$$(1.6) \quad Lu(x) = \text{SPV} \int_{y \neq x} (u(y) - u(x)) k_s(x, y) dy + \int_{y \neq x} (u(y) - u(x)) k_a(x, y) dy$$

for any  $u \in C_0^2(\mathbb{R}^n)$ , where  $C_0^2(\mathbb{R}^n)$  is the set of twice differentiable functions on  $\mathbb{R}^n$  with compact support.

*Proof. Step 1:* We first claim that  $C_0^2(\mathbb{R}^n) \subset C^*(\mathbb{R}^n)$ . Note that (1.4) implies (A0). For any  $n \geq 1$  and any increasing sequence of relatively compact and symmetric sets  $\{A_j\}_{j \in \mathbb{N}}$  such that  $\bigcup_{j \in \mathbb{N}} A_j = \mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$ , we find, by the Taylor formula and (A0), that for any  $j \geq 1$  and  $x \in \mathbb{R}^n$ ,

$$\begin{aligned} & \left| \int_{\{(x,y) \in A_j\}} (u(x) - u(y)) k_s(x, y) dy \right| \\ & \leq \int_{\{(x,y) \in A_j\}} \left( \left( \frac{1}{2} \|\nabla^2 u\|_\infty |x - y|^2 \right) \wedge (2 \|u\|_\infty) \right) k_s(x, y) dy \\ & \leq 2 \left( \|u\|_\infty \vee \|\nabla^2 u\|_\infty \right) \int_{\{(x,y) \in A_j\}} \left( 1 \wedge |x - y|^2 \right) k_s(x, y) dy \\ & \leq 2 \left( \|u\|_\infty \vee \|\nabla^2 u\|_\infty \right) \int \left( 1 \wedge |x - y|^2 \right) k_s(x, y) dy < \infty, \end{aligned}$$

where in the first inequality we have used the fact that

$$\int_{\{(x,y) \in A_j\}} \langle \nabla u(x), x - y \rangle k_s(x, y) dy = 0.$$

On the other hand, by the same reasoning as above, for any  $1 \leq j \leq m < \infty$ ,

$$\begin{aligned} & \left| \int_{\{(x,y) \in A_m \setminus A_j\}} (u(x) - u(y)) k_s(x, y) dy \right| \\ & \leq 2 \left( \|u\|_\infty \vee \|\nabla^2 u\|_\infty \right) \int_{\{(x,y) \in A_m \setminus A_j\}} \left( 1 \wedge |x - y|^2 \right) k_s(x, y) dy \xrightarrow{j, m \rightarrow \infty} 0. \end{aligned}$$

Thus, the limit  $\lim_{j \rightarrow \infty} \int_{\{(x,y) \in A_j\}} (u(x) - u(y)) k_s(x, y) dy$  exists. Furthermore, it is also easy to see from the arguments above that the limit is independent of the sequence  $\{A_j\}_{j \in \mathbb{N}}$ . Therefore, for any  $u \in C_0^2(\mathbb{R}^n)$ ,  $\text{SPV} \int_{y \neq x} (u(y) - u(x)) k_s(x, y) dy$  exists.

*Step 2:* The semi-Dirichlet form  $(\eta, \mathcal{F}^0)$  is a coercive closed form in the sense of Ma-Röckner, cf. [6, Chapter I, Definition 2.4, page 16]. Then, by [6, Chapter I, Proposition 2.16, page 23],  $(L, D(L))$  is the generator of the form  $(\eta, \mathcal{F}^0)$ , where  $D(L) = \{u \in \mathcal{F}^0 \mid v \mapsto \eta(u, v) \text{ is continuous with respect to } \|\cdot\|_{L^2} \text{ on } \mathcal{F}^0\}$ . According to Proposition 1.2, under (1.1), (1.3) and (1.4), the operator  $L$  is of the form (1.6) on the set  $C^*(\mathbb{R}^n) \cap D(L)$ . Therefore, it suffices to check that  $C_0^2(\mathbb{R}^n) \subset C^*(\mathbb{R}^n) \cap D(L)$ . Because of Step 1, we only need to verify that the operator  $L$  maps  $C_0^2(\mathbb{R}^n)$  into  $L^2(\mathbb{R}^n)$ .

Let  $I_x = \text{SPV} \int_{y \neq x} (u(y) - u(x)) k_s(x, y) dy$  and  $\mathbb{I}_x = \int_{y \neq x} (u(y) - u(x)) k_a(x, y) dy$ . By the Cauchy-Schwarz inequality,

$$\begin{aligned} \|\mathbb{I}_x\|_{L^2(dx)}^2 &= \int \left( \int_{y \neq x} (u(y) - u(x)) k_a(x, y) dy \right)^2 dx \\ &= \int \left( \int_{\{k_s(x, y) \neq 0\}} (u(y) - u(x)) \sqrt{k_s(x, y)} \frac{k_a(x, y)}{\sqrt{k_s(x, y)}} dy \right)^2 dx \\ &\leq \int \left( \int (u(y) - u(x))^2 k_s(x, y) dy \right) \left( \int_{\{k_s(x, y) \neq 0\}} \frac{k_a^2(x, y)}{k_s(x, y)} dy \right) dx \\ &\leq \left[ \sup_{x \in \mathbb{R}^n} \int_{\{k_s(x, y) \neq 0\}} \frac{k_a^2(x, y)}{k_s(x, y)} dy \right] \mathcal{E}(u, u) < \infty, \end{aligned}$$

where  $\frac{1}{2}\mathcal{E}(u, u)$  is the symmetric part of  $\eta(u, u)$  given by (1.2). On the other hand, for any  $r > 0$ , it holds that

$$\|I_x\|_{L^2(dx)}^2 \leq \|\mathbb{1}_{B_{2r}(0)}(x) I_x\|_{L^2(dx)}^2 + \|\mathbb{1}_{B_{2r}^c(0)}(x) I_x\|_{L^2(dx)}^2.$$

First,

$$\begin{aligned} \|\mathbb{1}_{B_{2r}(0)}(x) I_x\|_{L^2(dx)}^2 &= \int_{|x| \leq 2r} \left( \text{SPV} \int (u(y) - u(x)) k_s(x, y) dy \right)^2 dx \\ &\leq 2 (\|u\|_\infty \vee \|\nabla^2 u\|_\infty) \int_{|x| \leq 2r} \left( \int (1 \wedge |x - y|^2) k_s(x, y) dy \right)^2 dx < \infty. \end{aligned}$$

In the first inequality we have used again the fact that

$$\int_{\{(x, y) \in A\}} \langle \nabla u(x), x - y \rangle k_s(x, y) dy = 0$$

for any relatively compact and symmetric set  $A \subset \mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$ , and the last inequality follows from (1.4). Pick  $r > 1$  large enough such that  $\text{supp } u \subset B_r(0)$ . We get that

$$\begin{aligned} \|\mathbb{1}_{B_{2r}^c(0)}(x) I_x\|_{L^2(dx)}^2 &= \int_{|x| \geq 2r} \left( \text{SPV} \int (u(y) - u(x)) k_s(x, y) dy \right)^2 dx \\ &= \int_{|x| \geq 2r} \left( \int u(y) k_s(x, y) dy \right)^2 dx \\ &= \int_{|x| \geq 2r} \left( \int_{|y| \leq r} u(y) k_s(x, y) dy \right)^2 dx \\ &\leq \|u\|_\infty^2 \int \left( \int_{|x-y| > r} \mathbb{1}_{B_r(0)}(y) k_s(x, y) dy \right)^2 dx. \end{aligned}$$

If  $x \mapsto \int_{|y-x| \geq 1} k_s(x, y) dy \in L^2(dx)$ , then we have  $\|\mathbb{1}_{B_{2r}^c(0)}(x) I_x\|_{L^2(dx)}^2 < \infty$ .

If  $x \mapsto \int_{|y-x| \geq 1} k_s(x, y) dy \in L^\infty(dx)$ , then, by the Cauchy-Schwarz inequality,

$$\begin{aligned}
& \|u\|_\infty^2 \int \left( \int_{|x-y| > r} \mathbb{1}_{B_r(0)}(y) k_s(x, y) dy \right)^2 dx \\
& \leq \|u\|_\infty^2 \int \left( \int_{|x-y| > r} \mathbb{1}_{B_r(0)}(y) k_s(x, y) dy \right) \left( \int_{|x-y| > r} k_s(x, y) dy \right) dx \\
& \leq \|u\|_\infty^2 \left[ \sup_{x \in \mathbb{R}^n} \int_{|x-y| > r} k_s(x, y) dy \right] \iint_{|x-y| > r} \mathbb{1}_{B_r(0)}(y) k_s(x, y) dy dx \\
& = \|u\|_\infty^2 \left[ \sup_{x \in \mathbb{R}^n} \int_{|x-y| > r} k_s(x, y) dy \right] \int \mathbb{1}_{B_r(0)}(x) \int_{|x-y| > r} k_s(x, y) dy dx < \infty,
\end{aligned}$$

where in the equality above we have used the symmetry of  $k_s(x, y) dy dx$ , and the last inequality follows from (A0). This also gives us that  $\|\mathbb{1}_{B_{2r}^c(0)}(x) I_x\|_{L^2(dx)}^2 < \infty$ . The required assertion follows from all the conclusions above.  $\square$

## 2. THE ADJOINT OF A LÉVY TYPE OPERATOR ON $\mathbb{R}^n$

Assume that  $E = \mathbb{R}^n$  is equipped with the Euclidean metric  $d(x, y) = |x - y|$  and Lebesgue measure  $m(dx) = dx$  as reference measure. If  $k$  is symmetric, then  $k_a(x, y) = 0$  and Proposition 1.2 is identical with [7, Theorem 2.2]. In this case the symmetric Cauchy principle value integral in the representation of  $L$  can be rewritten as an absolutely convergent integral if we introduce a regularizing term in the integrand, see (2.7) below. This observation enables us to consider the (formal) adjoint of  $L$ .

Let  $C_0^\infty(\mathbb{R}^n)$  be the space of smooth functions with compact support on  $\mathbb{R}^n$ . For  $f \in C_0^\infty(\mathbb{R}^n)$ , define the following Lévy type operator

$$\begin{aligned}
(2.7) \quad Lf(x) &= \int \left( f(x+z) - f(x) - \nabla f(x) \cdot z \mathbb{1}_{\{|z| \leq 1\}} \right) j(x, x+z) dz \\
&\quad + \frac{1}{2} \int_{|z| \leq 1} z (j(x, x+z) - j(x, x-z)) dz \cdot \nabla f(x),
\end{aligned}$$

where  $\int_{z \neq 0} (1 \wedge |z|^2) j(x, x+z) dz < \infty$  and

$$\int_{|z| \leq 1} |z| |j(x, x+z) - j(x, x-z)| dz < \infty$$

for all  $x \in \mathbb{R}^n$ .

We will now present an explicit expression of the (formal) adjoint of the operator  $L$ . To state our result, we need a few assumptions. As before we write  $j_s$  and  $j_a$  for the symmetric and antisymmetric parts of  $j$ , i.e.

$$j_s(x, y) := \frac{1}{2} (j(x, y) + j(y, x)) \quad \text{and} \quad j_a(x, y) := \frac{1}{2} (j(x, y) - j(y, x)).$$

For  $x, z \in \mathbb{R}^n$ , we denote by

$$j^*(x, z) := |j(x, x+z) - j(x, x-z)| + |j(x+z, x) - j(x-z, x)|.$$



- (H1)  $x \mapsto \int (1 \wedge (y-x)^2) j_s(x, y) dy \in L^2_{\text{loc}}(dx);$
- (H2)  $x \mapsto \int_{|y-x| \geq 1} j_s(x, y) dy \in L^2(dx) \cup L^\infty(dx);$
- (H3)  $x \mapsto \int_{|z| \leq 1} |z| j^*(x, z) dz \in L^2_{\text{loc}}(dx);$
- (H4)  $\sup_{x \in \mathbb{R}^n} \int_{j_s(x, y) \neq 0} \frac{j_a(x, y)^2}{j_s(x, y)} dy < \infty;$
- (H5)  $\sup_{x \in K} \sup_{\epsilon > 0} \left| \int_{|y-x| \geq \epsilon} j_a(x, y) dy \right| < \infty$  for every compact set  $K \subset \mathbb{R}^n$ .

Note that (H1) is just (1.4), which implies (A0). (H2) is (1.5), and (H4) is the same as (1.1). The conditions (H3) and (H5) have no direct counterpart in Section 1. It is clear that (H3) implies

$$(2.8) \quad x \mapsto \int_{|z| \leq 1} |z| |j_s(x, x+z) - j_s(x, x-z)| dz \in L^2_{\text{loc}}(dx).$$

As in the proof of Theorem 1.3, we can easily obtain that under (H1)–(H3) the operator  $L$  given by (2.7) maps  $C_0^\infty(\mathbb{R}^n)$  into  $L^2(\mathbb{R}^n)$ .

**Theorem 2.1.** *Let  $L$  be the operator given by (2.7) and assume that (H1)–(H5) hold. Then the formal adjoint  $L^*$  is for any  $f \in C_0^\infty(\mathbb{R}^n)$*

$$(2.9) \quad \begin{aligned} L^* f(x) = & \int (f(x+z) - f(x) - \nabla f(x) \cdot z \mathbf{1}_{\{|z| \leq 1\}}) j(x+z, x) dz \\ & + \frac{1}{2} \int_{|z| \leq 1} z (j(x+z, x) - j(x-z, x)) dz \cdot \nabla f(x) + f(x) \kappa(dx), \end{aligned}$$

where  $\kappa(dx)$  is a (signed) Radon measure on  $\mathbb{R}^n$  which is the vague limit of the sequence of (signed) measures  $(4 \int_{|x-y| > 1/m} j_a(x, y) dy dx)_{m \in \mathbb{N}}$ .

By (H1), (2.8) and (H5), the operator  $L^*$  given by (2.9) is well defined on  $C_0^\infty(\mathbb{R}^n)$ . From the proof of Theorem 1.3, we know that, under (H1)–(H3) and (H5), the operator  $L^*$  maps  $C_0^\infty(\mathbb{R}^n)$  into  $L^2(\mathbb{R}^n)$ . In particular, if  $j$  is symmetric, i.e.  $j(x, y) = j(y, x)$  for all  $x, y \in \mathbb{R}^n$ , Theorem 2.1 shows that  $L = L^*$  on  $C_0^\infty(\mathbb{R}^n)$ . That is,  $L$  defined by (2.7) is a symmetric Lévy type operator. For further details we refer to [7, Theorem 2.2] and [9, Theorem 1.2].

*Proof of Theorem 2.1.* The proof is divided into four steps. Throughout the proof we fix some  $f, g \in C_0^\infty(\mathbb{R}^n)$ .

*Step 1:* For any  $\varepsilon > 0$ , we define

$$L_\varepsilon f(x) := \int_{|z| \geq \varepsilon} (f(x+z) - f(x) - \nabla f(x) \cdot z \mathbf{1}_{\{|z| \leq 1\}}) j(x, x+z) dz$$

$$\begin{aligned}
& + \frac{1}{2} \int_{\varepsilon \leq |z| \leq 1} z (j(x, x+z) - j(x, x-z)) dz \cdot \nabla f(x) \\
& = \int_{|z| \geq \varepsilon} (f(x+z) - f(x)) j(x, x+z) dz \\
& \quad - \frac{1}{2} \int_{\varepsilon \leq |z| \leq 1} z (j(x, x+z) + j(x, x-z)) dz \cdot \nabla f(x).
\end{aligned}$$

Since for every  $x \in \mathbb{R}^n$ ,

$$\int_{\varepsilon \leq |z| \leq 1} z (j(x, x+z) + j(x, x-z)) dz = 0,$$

we get

$$L_\varepsilon f(x) = \int_{|z| \geq \varepsilon} (f(x+z) - f(x)) j(x, x+z) dz;$$

due to (H1),  $L_\varepsilon f \in L^2(\mathbb{R}^n)$ . Consequently,

$$\begin{aligned}
\langle L_\varepsilon f, g \rangle &= \int g(x) \int_{|z| \geq \varepsilon} (f(x+z) - f(x)) j(x, x+z) dz dx \\
&= \int g(x) \int_{|y-x| \geq \varepsilon} (f(y) - f(x)) j(x, y) dy dx.
\end{aligned}$$

*Step 2:* We will now show that the limit

$$(2.10) \quad A(f, g) := -\frac{1}{2} \lim_{\varepsilon \rightarrow 0} \left[ \langle L_\varepsilon f, g \rangle + \langle f, L_\varepsilon g \rangle \right]$$

exists. Observe that

$$\begin{aligned}
& \langle L_\varepsilon f, g \rangle + \langle f, L_\varepsilon g \rangle \\
&= \left[ \int g(x) \int_{|y-x| \geq \varepsilon} (f(y) - f(x)) j(x, y) dy dx \right. \\
& \quad \left. + \int f(x) \int_{|y-x| \geq \varepsilon} (g(y) - g(x)) j(x, y) dy dx \right] \\
&= \iint_{|y-x| \geq \varepsilon} \left( (f(y) - f(x))g(x) + (g(y) - g(x))f(x) \right) j(x, y) dx dy \\
&= \iint_{|y-x| \geq \varepsilon} \left( (f(y) - f(x))g(x) + (g(y) - g(x))f(x) \right) j_s(x, y) dx dy \\
& \quad + \iint_{|y-x| \geq \varepsilon} \left( (f(y) - f(x))g(x) + (g(y) - g(x))f(x) \right) j_a(x, y) dx dy \\
&=: I_1^\varepsilon + I_2^\varepsilon.
\end{aligned}$$

If we change  $x$  and  $y$  in the expression of  $I_1^\varepsilon$ , we get

$$I_1^\varepsilon = \iint_{|y-x| \geq \varepsilon} \left( (f(y) - f(x))g(y) + (g(y) - g(x))f(y) \right) j_s(x, y) dx dy,$$

which, if added to the original expression for  $I_1^\varepsilon$ , yields that

$$I_1^\varepsilon = \iint_{|y-x| \geq \varepsilon} (f(y) - f(x))(g(y) - g(x))j_s(x, y) dx dy.$$

Because of (H1) we find

$$(2.11) \quad I_1 := \lim_{\varepsilon \rightarrow 0} I_1^\varepsilon = \iint_{x \neq y} (f(y) - f(x))(g(y) - g(x))j_s(x, y) dx dy.$$

On the other hand, we see as in the proof of Theorem 1.1, that under (H1) and (H4), the limit  $I_2 := \lim_{\varepsilon \rightarrow 0} I_2^\varepsilon$  exists and

$$(2.12) \quad I_2 = \iint \left( (f(y) - f(x))g(x) + (g(y) - g(x))f(x) \right) j_a(x, y) dx dy$$

with an absolutely convergent integral.

*Step 3:* According to [7, Theorem 2.2], the assumptions (H1) and (H3) imply that

$$(2.13) \quad \frac{1}{2} I_1 = -\langle \tilde{L}f, g \rangle,$$

where

$$\begin{aligned} \tilde{L}f(x) &:= \int_{|z| > 0} \left( f(x+z) - f(x) - \nabla f(x) \cdot z \mathbf{1}_{\{|z| \leq 1\}} \right) j_s(x, x+z) dz \\ &\quad + \frac{1}{2} \int_{0 < |z| \leq 1} z (j_s(x, x+z) - j_s(x, x-z)) dz \cdot \nabla f(x). \end{aligned}$$

As the same reasoning as above, the proof of Theorem 1.3 shows that, under (H1)—(H3), the operator  $\tilde{L}$  maps  $C_0^\infty(\mathbb{R}^n)$  into  $L^2(\mathbb{R}^n)$ .

If we change  $x$  and  $y$  in the expression of  $I_2$ , we get because of the antisymmetry of  $j_a$

$$I_2 = \iint_{x \neq y} \left( (f(y) - f(x))g(y) + (g(y) - g(x))f(y) \right) j_a(x, y) dx dy$$

and if we add this to the original expression for  $I_2$  we see

$$I_2 = 2 \iint_{x \neq y} \left( f(x)g(x) - f(y)g(y) \right) j_a(x, y) dx dy.$$

In the same way we find that

$$\begin{aligned} I_2^\varepsilon &= 2 \iint_{|y-x| \geq \varepsilon} \left( f(x)g(x) - f(y)g(y) \right) j_a(x, y) dx dy \\ &= 2 \iint_{|y-x| \geq \varepsilon} f(x)g(x) j_a(x, y) dx dy - 2 \iint_{|y-x| \geq \varepsilon} f(y)g(y) j_a(x, y) dx dy \\ &= 2 \iint_{|y-x| \geq \varepsilon} f(x)g(x) j_a(x, y) dx dy + 2 \iint_{|y-x| \geq \varepsilon} f(x)g(x) j_a(x, y) dx dy \\ &= 4 \int f(x)g(x) \left[ \int_{|y-x| \geq \varepsilon} j_a(x, y) dy \right] dx. \end{aligned}$$

Since  $f, g \in C_0^\infty(\mathbb{R}^n)$  are arbitrary and since the limit  $\lim_{\varepsilon \rightarrow 0} I_2^\varepsilon = I_2$  exists, we see that the limit

$$\kappa = \lim_{\varepsilon \rightarrow 0} \left[ 4 \int_{|y-x| \geq \varepsilon} j_a(x, y) dy \right]$$

exists in the sense of distributions and defines an element  $\kappa \in \mathcal{D}'$  (the dual space of  $C_c^\infty(\mathbb{R}^n)$ ), and (H5) shows that  $\kappa$  is a distribution of order zero—i.e.  $|\kappa(u)| \leq c_K \|u\|_\infty$  for all continuous functions  $u$  with support in the compact set  $K$ —, hence a (signed) Radon measure.

Thus, by (H5) again, for all  $f, g \in C_0^\infty(\mathbb{R}^n)$ ,

$$(2.14) \quad \frac{1}{2} I_2 = \frac{1}{2} \int f(x) g(x) \kappa(dx) := \langle \widehat{L} f, g \rangle.$$

*Step 4:* The formal adjoint  $L^*$  of the operator  $L$  is defined by

$$\langle L^* f, g \rangle = \langle f, L g \rangle.$$

Note that under (H1)–(H3), for any  $f \in C_0^\infty(\mathbb{R}^n)$ ,

$$\lim_{\varepsilon \rightarrow 0} \|L_\varepsilon f - L f\|_{L^2} = 0.$$

Therefore, for any  $f, g \in C_0^\infty(\mathbb{R}^n)$ ,

$$(2.15) \quad A(f, g) = -\frac{1}{2} \lim_{\varepsilon \rightarrow 0} \left[ \langle L_\varepsilon f, g \rangle + \langle f, L_\varepsilon g \rangle \right] = -\frac{1}{2} \left[ \langle L f, g \rangle + \langle f, L g \rangle \right].$$

Combining (2.10)–(2.15), we have

$$\langle L^* f, g \rangle + \langle L f, g \rangle = \langle f, L g \rangle + \langle L f, g \rangle = 2 \left( \langle \widetilde{L} f, g \rangle + \langle \widehat{L} f, g \rangle \right).$$

Therefore,

$$L^* = 2(\widetilde{L} + \widehat{L}) - L$$

which is what we have claimed.  $\square$

### 3. EXAMPLE: STABLE-LIKE PROCESSES

Let  $E = \mathbb{R}^n$ , and  $m(dx) = dx$  be Lebesgue measure on  $\mathbb{R}^n$ . Consider the following integro-differential operator

$$Lu(x) = w(x) \int_{z \neq 0} \left( u(x+z) - u(x) - \nabla u(x) \cdot z \mathbb{1}_{\{|z| \leq 1\}}(z) \right) |z|^{-n-\alpha(x)} dz$$

for  $u \in C_0^\infty(\mathbb{R}^n)$ . The weight function  $w(x)$  is chosen in such a way that

$$w(x) = \alpha(x) 2^{\alpha(x)-1} \Gamma((\alpha(x) + n)/2) / \left( \pi^{n/2} \Gamma(1 - \alpha(x)/2) \right),$$

and so  $Le_\xi(x) = -|\xi|^{\alpha(x)} e_\xi(x)$ , where  $e_\xi(x) = e^{ix \cdot \xi}$ , see e.g. [2, Exercise 18.23, page 184]. With this norming,  $L$  can be written as a pseudo-differential operator  $-p(x, D)$  with the symbol  $-|\xi|^{\alpha(x)}$ ,

$$Lu(x) = \int e^{ix \cdot \xi} |\xi|^{\alpha(x)} \widehat{u}(\xi) d\xi = -(-\Delta)^{\alpha(x)} u(x),$$

and this shows that  $L = -(-\Delta)^{\alpha(x)}$  is a stable-like operator in the sense of Bass [1].

For  $r > 0$ , define

$$\beta(r) := \sup_{|x-y| \leq r} |\alpha(x) - \alpha(y)|.$$

**Proposition 3.1.** *Let  $L = -(-\Delta)^{\alpha(x)}$  and suppose that there exist  $0 < \alpha_1 \leq \alpha_2 < 2$  such that*

$$\alpha_1 \leq \alpha(x) \leq \alpha_2 \quad \text{for all } x \in \mathbb{R}^n,$$

and

$$\int_0^1 \frac{(\beta(r) |\log r|)^2}{r^{1+\alpha_2}} dr < \infty.$$

(i) *The operator  $(L, C_0^\infty(\mathbb{R}^n))$  generates a regular lower bounded semi-Dirichlet form on  $L^2(\mathbb{R}^n)$  associated with the kernel*

$$k(x, y) = w(x) |x - y|^{-n-\alpha(x)}.$$

(ii) *If for all compact sets  $K \subset \mathbb{R}^n$*

$$\sup_{x \in K} \sup_{\epsilon > 0} \left| \int_{|z| \geq \epsilon} \left( \frac{w(x)}{|z|^{n+\alpha(x)}} - \frac{w(x+z)}{|z|^{n+\alpha(x+z)}} \right) dz \right| \leq c_K < \infty,$$

then the formal adjoint of  $L$  is given by

$$\begin{aligned} L^* f(x) &= \int \left( f(x+z) - f(x) - \nabla f(x) \cdot z \mathbf{1}_{\{|z| \leq 1\}} \right) \frac{w(x+z)}{|z|^{n+\alpha(x+z)}} dz \\ &\quad + \frac{1}{2} \int_{\{|z| \leq 1\}} z \left( \frac{w(x+z)}{|z|^{n+\alpha(x+z)}} - \frac{w(x-z)}{|z|^{n+\alpha(x-z)}} \right) dz \cdot \nabla f(x) + f(x) \kappa(dx). \end{aligned}$$

where  $\kappa(dx)$  is a (signed) Radon measure on  $\mathbb{R}^n$  which is the vague limit of the sequence of (signed) measures  $(2 \int_{|z| > 1/m} \left( \frac{w(x)}{|z|^{n+\alpha(x)}} - \frac{w(x+z)}{|z|^{n+\alpha(x+z)}} \right) dz dx)_{m \in \mathbb{N}}$ .

*Proof.* We check the conditions of Theorems 1.1 and 2.1. Set

$$k(x, y) = w(x) |x - y|^{-n-\alpha(x)}.$$

Then,

$$\begin{aligned} k_s(x, y) &= \frac{1}{2} \left( w(x) |x - y|^{-n-\alpha(x)} + w(y) |x - y|^{-n-\alpha(y)} \right), \\ k_a(x, y) &= \frac{1}{2} \left( w(x) |x - y|^{-n-\alpha(x)} - w(y) |x - y|^{-n-\alpha(y)} \right). \end{aligned}$$

From the definition of  $w(x)$  it is easy to see that there exist constants  $c_j > 0$ ,  $j = 1, 2, 3$ , such that for any  $x, y \in \mathbb{R}^n$ ,

$$c_1 \leq w(x) \leq c_2, \quad |w(x) - w(y)| \leq c_3 |\alpha(x) - \alpha(y)|,$$

see [3, proof of Proposition 3].

(i) Since (A0) is obviously satisfied, we only have to verify (1.1) of Theorem 1.1. We have

$$\sup_{x \in \mathbb{R}^n} \int_{|x-y| \geq 1} \frac{k_a^2(x, y)}{k_s(x, y)} dy \leq \sup_{x \in \mathbb{R}^n} \int_{|x-y| \geq 1} k_s(x, y) dy \leq c \int_1^\infty r^{-1-\alpha_1} dr < \infty.$$

To see

$$\sup_{x \in \mathbb{R}^n} \int_{|x-y| \leq 1} \frac{k_a^2(x, y)}{k_s(x, y)} dy < \infty$$

we write

$$k_a(x, y) = \frac{1}{2} \left[ (w(x) - w(y)) |x - y|^{-n-\alpha(x)} + w(y) |x - y|^{-n} (|x - y|^{-\alpha(x)} - |x - y|^{-\alpha(y)}) \right].$$

Then

$$\begin{aligned} & \int_{|x-y| \leq 1} \frac{(w(x) - w(y))^2 (|x - y|^{-n-\alpha(x)})^2}{w(x) |x - y|^{-n-\alpha(x)} + w(y) |x - y|^{-n-\alpha(y)}} dy \\ & \leq c \int_{|x-y| \leq 1} \frac{(\alpha(x) - \alpha(y))^2}{|x - y|^{n+\alpha(x)}} dy \\ & \leq c \int_{|x-y| \leq 1} \frac{\beta^2(|x - y|)}{|x - y|^{n+\alpha_2}} dy \\ & = c \int_0^1 \frac{\beta^2(r)}{r^{1+\alpha_2}} dr. \end{aligned}$$

Since

$$|x - y|^{-\alpha(x)} - |x - y|^{-\alpha(y)} = \int_{\alpha(y)}^{\alpha(x)} |x - y|^{-u} \log |x - y|^{-1} du,$$

we obtain for all  $|x - y| \leq 1$ ,

$$(|x - y|^{-\alpha(x)} - |x - y|^{-\alpha(y)})^2 \leq (\log |x - y|^{-1})^2 (\alpha(x) - \alpha(y))^2 |x - y|^{-2(\alpha(x) \vee \alpha(y))}.$$

Therefore,

$$\begin{aligned} & \int_{|x-y| \leq 1} \frac{w(y)^2 |x - y|^{-2n} (|x - y|^{-\alpha(x)} - |x - y|^{-\alpha(y)})^2}{w(x) |x - y|^{-n-\alpha(x)} + w(y) |x - y|^{-n-\alpha(y)}} dy \\ & \leq c \int_{|x-y| \leq 1} \frac{(\alpha(x) - \alpha(y))^2 (\log |x - y|^{-1})^2}{|x - y|^n} \frac{|x - y|^{-2(\alpha(x) \vee \alpha(y))}}{w(x) |x - y|^{-\alpha(x)} + w(y) |x - y|^{-\alpha(y)}} dy \\ & \leq c \int_{|x-y| \leq 1} \frac{(\alpha(x) - \alpha(y))^2 (\log |x - y|^{-1})^2}{|x - y|^n} |x - y|^{-(\alpha(x) \vee \alpha(y))} dy \\ & \leq c \int_{|x-y| \leq 1} \frac{\beta^2(|x - y|) (\log |x - y|^{-1})^2}{|x - y|^{n+\alpha_2}} dy \\ & \leq c \int_0^1 \frac{(\beta(r) |\log r|)^2}{r^{1+\alpha_2}} dr, \end{aligned}$$

and (1.1) follows.

(ii) Clearly, the conditions (H1) and (H2) are satisfied. From part (i), we know that (H4) is also valid. Therefore, by Theorem 2.1 and the assumption, we have to verify (H3). For all  $x \in \mathbb{R}^n$ ,

$$\int_{|x-y| \leq 1} |x - y| \left| \frac{w(y)}{|x - y|^{n+\alpha(y)}} - \frac{w(x)}{|x - y|^{n+\alpha(x)}} \right| dy$$

$$\begin{aligned}
&\leq \int_{|x-y| \leq 1} |x-y| |w(x) - w(y)| |x-y|^{-n-\alpha(y)} dy \\
&\quad + \int_{|x-y| \leq 1} |x-y| w(x) \left| |x-y|^{-n-\alpha(y)} - |x-y|^{-n-\alpha(y)} \right| dy.
\end{aligned}$$

With similar arguments as in the proof of part (i) we see that the right hand side of the inequality above is smaller than

$$c \int_0^1 \frac{\beta(r)(1 + |\log r|)}{r^{\alpha_2}} dr \leq c' \left( \int_0^1 \frac{\beta(r) |\log r|}{r^{\alpha_2}} dr + 1 \right).$$

Pick  $\gamma < 1/2$  such that  $\alpha_2 \leq 1 + 2\gamma < 2$ . By the Cauchy-Schwarz inequality, we find

$$\begin{aligned}
\int_0^1 \frac{\beta(r) |\log r|}{r^{\alpha_2}} dr &\leq \left( \int_0^1 \frac{1}{r^{2\gamma}} dr \right)^{1/2} \left( \int_0^1 \frac{\beta(r)^2 |\log r|^2}{r^{2\alpha_2-2\gamma}} dr \right)^{1/2} \\
&\leq \frac{1}{\sqrt{1-2\gamma}} \left( \int_0^1 \frac{\beta(r)^2 |\log r|^2}{r^{1+\alpha_2}} dr \right)^{1/2}. \quad \square
\end{aligned}$$

We close with this section with some comments on related results in [3, 8] and our Proposition 3.1.

**Remark 3.2.** (i) Assume that for  $r \rightarrow 0$

$$\beta(r) \asymp r^\beta, \quad \beta > \alpha_2/2, \quad \text{or} \quad \beta(r) \asymp r^{\alpha_2/2} |\log r|^\varepsilon, \quad \varepsilon < -3/2.$$

Then Proposition 3.1(i) applies and shows that the operator  $(L, C_0^\infty(\mathbb{R}^n))$  generates a regular lower bounded semi-Dirichlet form on  $L^2(\mathbb{R}^n)$ . This is, in particular, the case if the index function  $\alpha(x)$  is locally Lipschitz continuous. Thus, Proposition 3.1(i) improves [3, Proposition 3] where the following assumptions are used: *there exist positive constants  $\alpha_1, \alpha_2, M$  and  $\delta$  such that for  $x, y \in \mathbb{R}^n$ ,*

$$0 < \alpha_1 \leq \alpha(x) \leq \alpha_2 < 2 \quad \text{with} \quad \alpha_2 < 1 + \alpha_1/2$$

and

$$|\alpha(x) - \alpha(y)| \leq M|x - y|^\delta \quad \text{with} \quad 0 < \frac{1}{2}(2\alpha_2 - \alpha_1) < \delta \leq 1.$$

(ii) With essentially the same calculations as in the proof of Proposition 3.1(ii) we can see that

$$\int_0^1 \frac{\beta(r) |\log r|}{r^{1+\alpha_2}} dr < \infty$$

guarantees that

$$\int \left( \frac{w(x)}{|z|^{n+\alpha(x)}} - \frac{w(x+z)}{|z|^{n+\alpha(x+z)}} \right) dz$$

exists in the usual sense. This is, for example, the case if *there exist positive constants  $\alpha_1, \alpha_2, \delta$  and  $M$  such that  $0 < \alpha_1 \leq \alpha_2 < 1$ ,  $\delta \in (\alpha_2, 1]$ , and for all  $x, y \in \mathbb{R}^n$ ,*

$$\alpha_1 \leq \alpha(x) \leq \alpha_2 \quad \text{and} \quad |\alpha(x) - \alpha(y)| \leq M|x - y|^\delta.$$

This shows that Proposition 3.1(ii) covers the conclusion in [8, Remark 4].

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#### REFERENCES

- [1] Bass, R.: Uniqueness in law for pure jump Markov processes, *Probab. Theory Related Fields* **79** (1988), 271–287.
- [2] Berg, C. and Forst, G.: *Potential Theory on Locally Compact Abelian Groups*, Springer-Verlag, Ergebnisse Math. Grenzgeb. vol. **87**, Berlin 1975.
- [3] Fukushima, M. and Uemura, T.: Jump-type Hunt processes generated by lower bounded semi-Dirichlet forms, to appear in *Ann. Probab.* (2011).
- [4] Fukushima, M., Oshima, Y. and Takeda, M.: *Dirichlet Forms and Symmetric Markov Processes*, de Gruyter Studies in Math. vol. **19**, Walter de Gruyter, Berlin 1994.
- [5] Hu, Z.C., Ma, Z.M. and Sun, W.: Extensions of Lévy-Khintchine formula and Beurling-Deny formula in semi-Dirichlet forms setting, *J. Funct. Anal.* **239** (2006), 179–213.
- [6] Ma, Z.M. and Röckner, M.: *Introduction to the Theory of (Non-Symmetric) Dirichlet Forms*, Universitext, Springer-Verlag, Berlin 1992.
- [7] Schilling, R.L. and Uemura, T.: On the Feller property of Dirichlet forms generated by pseudo differential operators, *Tohoku Math. J.* **59** (2007), 401–422.
- [8] Uemura, T.: A remark on non-local operators with variable order, *Osaka J. Math.* **46** (2009), 503–514.
- [9] Wang, J.: Symmetric Lévy type operator, *Acta Math. Sin. Eng. Ser.* **25** (2009), 39–46.